Math 120A
Differential Geometry

Sample Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a smooth curve. Define a reparametrization $\widetilde{\gamma}$ of γ.

> **Solution:** We say that $\widetilde{\gamma} : (\widetilde{\alpha}, \widetilde{\beta}) \to \mathbb{R}^n$ is a reparametrization of γ if there is a smooth bijection $\phi : (\widetilde{\alpha}, \widetilde{\beta}) \to (\alpha, \beta)$ with smooth inverse such that $\gamma \circ \phi = \widetilde{\gamma}$.

(b) [5pts.] Show that reparametrization is transitive: if $\widetilde{\gamma}$ is a reparametrization of γ, and $\widehat{\gamma}$ is a reparametrization of $\widetilde{\gamma}$, then $\widehat{\gamma}$ is a reparametrization of γ.

> **Solution:** Say $\widetilde{\gamma} \circ \phi = \gamma$ and $\widehat{\gamma} \circ \psi = \widetilde{\gamma}$, for $\phi : (\widetilde{\alpha}, \widetilde{\beta}) \to (\alpha, \beta)$ and $\psi : (\widehat{\alpha}, \widehat{\beta}) \to (\widetilde{\alpha}, \widetilde{\beta})$ both smooth and smoothly invertible bijections. Then $\gamma \circ (\phi \circ \psi) = \widehat{\gamma}$, and $\phi \circ \psi$ is also a smooth bijection with smooth inverse $\psi^{-1} \circ \phi^{-1}$, so $\widehat{\gamma}$ is a reparametrization of γ.

Problem 2.

(a) [5pts.] Give a formula for the curvature of (i) a unit speed curve and (ii) an arbitrary regular curve in $\mathbb{R}^3$.

> **Solution:** The curvature of $\gamma(t)$ unit-speed is $\kappa = ||\ddot{\gamma}(t)||$. For an arbitrary regular curve, we have
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> $$\kappa = \frac{||\ddot{\gamma}(t) \times \dot{\gamma}(t)||}{||\dot{\gamma}(t)||^3}$$

(b) [5pts.] Let $\gamma(s)$ be a unit speed plane curve with signed curvature function $k(s)$. Let $\gamma_a(s) = a\gamma(s)$ be the image of γ under the dilation $\mathbf{v} \to a\mathbf{v}$, for a a positive constant. Prove that the signed curvature of γ_a (expressed in terms of *its* arclength) is $\frac{1}{a}k(\frac{s}{a})$.

> **Solution:** Note that $\dot{\gamma}_a(s) = a\dot{\gamma}(s)$ has length a for all s. So the arclength parameter of γ_a is $q = \frac{s}{a}$, and in particular $s \to a\gamma(\frac{s}{a})$ is a unit speed reparametrization of γ_a with unit tangent vector $\mathbf{t} = \dot{\gamma}(\frac{s}{a})$. The acceleration vector of this unit-speed reparametrization is $\frac{1}{a}\ddot{\gamma}(\frac{s}{a})$. Since γ_a has the same signed unit normal as γ, we conclude the curvature function of γ_a is $\frac{1}{a}k(\frac{s}{a})$.
> Heavy chain rule version: We're interested in $\frac{d}{dq}\mathbf{t}(q) = \frac{ds}{dq}\frac{d}{ds}(\mathbf{t}(q) = a\frac{d}{ds}(\gamma'(\frac{s}{a})) = a(\frac{1}{a})\gamma''(\frac{s}{a}) = \gamma''(\frac{s}{a}) = \frac{1}{a}\kappa_s\mathbf{n_s}$, where the last $\frac{1}{a}$ comes from the fact that $\frac{s}{a}$ is not the arc speed parameter of γ, s is.
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> (Alternately, you can argue that the curvature clearly has the same sign as the signed curvature of γ, and then use the formula from part (a).)

Problem 3.

(a) [5pts.] Give a formula for the torsion of (i) a unit speed curve in $\mathbb{R}^3$ and (ii) a regular curve in $\mathbb{R}^3$. Do not forget to mention any hypotheses you need for your formula to make sense.

> **Solution:** Let γ be a unit speed curve with everywhere nonzero curvature κ, $\mathbf{t}$ be its unit tangent vector, and $\mathbf{n} = \frac{1}{\kappa}\dot{\mathbf{t}}$ be its preferred unit normal. Then if $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the binormal vector, τ is the number satisfying $\dot{\mathbf{b}} = -\tau\mathbf{n}$. More generally, if γ is an arbitrary regular curve with everywhere nonzero curvature, its torsion is given by
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> $$\frac{\dddot{\gamma} \cdot (\dot{\gamma} \times \ddot{\gamma})}{||\ddot{\gamma} \times \dot{\gamma}||^2}$$

(b) [5pts.] Show that if γ is a regular curve with $\tau \equiv 0$ (and defined everywhere), then γ lies in a plane. [Hint: Recall that the equation of a plane is $\mathbf{x} \cdot \mathbf{v} = d$, where $\mathbf{v}$ is a constant vector and d is a scalar. What vector do you expect $\mathbf{v}$ to be?]

> **Solution:** Without loss of generality, γ is unit-speed. We guess that the correct plane is $\mathbf{x} \cdot \mathbf{b} = d$. It suffices to show the derivative of $\gamma\dot{\mathbf{b}}$ is zero. But this derivative is $\dot{\gamma} \cdot \mathbf{b} + \gamma \cdot \dot{\mathbf{b}} = \mathbf{t} \cdot \mathbf{b} + 0 = 0$ since $\mathbf{b}$ and $\mathbf{t}$ are orthogonal. We conclude that $\gamma \cdot \mathbf{b}$ is constant, so γ lies in a plane $\mathbf{x} \cdot \mathbf{b} = d$.

Problem 4.

Let $\gamma : (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve.

(a) [5pts.] Define the arclength of $\gamma(t)$.

> **Solution:** The arc length is $s(t) = \int_{t_0}^{t} ||\dot{\gamma}(u)||du$, where t_0 is some starting parameter.

(b) [5pts.] Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a regular T-periodic curve, and let ℓ be the length of the curve over $[0, T]$. Prove that a unit-speed reparametrization of γ is ℓ-periodic.

> **Solution:** Without loss of generality we can choose 0 as the starting point for arc length. Let $\widetilde{\gamma}(s)$ be the arclength reparametrization, which is unit speed. We know that $\gamma(t+T) = \gamma(t)$ for all T, so it follows that $\dot{\gamma}(t) = \dot{\gamma}(t+T)$. Therefore $s(t+T) = \int_0^{t+T} ||\dot{\gamma}(u)||du = \int_0^t ||\dot{\gamma}(u)||du + \int_t^{t+T} ||\dot{\gamma}(u)||du = s(t) + \ell$. Therefore since $\gamma(t+T) = \gamma(t)$, we see $\widetilde{\gamma}(s(t)) = \widetilde{\gamma}(s(t) + \ell)$. Since T was the smallest number for which the first equality is true, we see ℓ is the smallest number for which the second is true, so ℓ is the period of $\widetilde{\gamma}$. All other unit-speed

parameters for γ are $\pm s + c$, with c a constant; reparametrizing by any of these stays ℓ-periodic.

Problem 5.
Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a unit-speed curve of nowhere-vanishing curvature, and $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the orthonormal system consisting of the unit tangent, normal, and binormal vector functions. Suppose that $\mathbf{t}$ makes a fixed angle with a constant unit vector $\mathbf{a}$, i.e. $\mathbf{t} \cdot \mathbf{a} = \cos\theta$, for θ constant.

(a) [3pts.] Prove that $\mathbf{n} \cdot \mathbf{a} = 0$.

Solution: We differentiate the relationship $\mathbf{t} \cdot \mathbf{a} = \cos\theta$, getting $(\kappa\mathbf{n}) \cdot \mathbf{a} + \mathbf{n} \cdot 0 = 0$, so since $\kappa \neq 0$, we get $\mathbf{n} \cdot \mathbf{a} = 0$.

(b) [3pts.] Prove that $\mathbf{a}$ can be written as a linear combination $\mathbf{a} = \lambda\mathbf{t} + \mu\mathbf{b}$. What are the coefficients? (Keep in mind that $\mathbf{a}$ is a unit vector.)

Solution: We know $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthornormal basis for $\mathbb{R}^3$, and $\mathbf{a} \perp \mathbf{n}$, so we must have $\mathbf{a} = \lambda\mathbf{t} + \mu\mathbf{b}$. Since $\mathbf{a} \cdot \mathbf{t} = \cos\theta$, we know $\lambda = \cos\theta$. But $\mathbf{a}$ is a unit vector, so $\mu = \pm\sin\theta$.

(c) [4pts.] Show that if κ and τ are the torsion and curvature of γ, then $\tau = \pm\kappa\cot\theta$. [Hint: What is the derivative of the relationship you found in part (b)?]

Solution: Differentiating the relationship $\mathbf{a} = \cos\theta\mathbf{t} \pm \sin\theta\mathbf{b}$, we see that $0 = -\sin\theta(\kappa\mathbf{n}) \pm \cos\theta(-\tau\mathbf{n})$. We conclude $\kappa\sin\theta = \pm\tau\cos\theta$, implying that $\tau = \pm\kappa\cot\theta$.